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# Some eigenvalue distribution functions of the Laguerre ensemble

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**Abstract.** In this paper we revisit the smallest-eigenvalue distribution of the Laguerre ensemble by presenting in closed form certain previously obtained integrals. With this information we compute, using Dyson's continuum approximation, the two-smallest-eigenvalue distribution of the Laguerre ensemble. Higher-order contributions to the free energies describing these two probabilities in certain scaling limits for $\beta \neq 2$, which can be interpreted in this context as entropic effect, are found.

## 1. Introduction

In a previous paper [1] the smallest-eigenvalue distribution function for the Laguerre ensemble was found via Dyson's [5] continuum approximation in a certain (double-) scaling limit (to be described below). The continuum approximation treats the eigenvalues of an ensemble of $N \times N$ random matrices by a continuous fluid in the limit of large $N$, with its thermodynamic free energy completely characterized as a functional of the fluid density. Although a rigorous proof of the general validity of such an approximation is not available, we have confidence in the robustness of this approach, since results obtained using it [5, 1–4] compare favourably with those obtained from exact analyses, whenever such comparison are available.

Quite generally, the probability functions which concern us can be computed as the change in the free energy when the support of the fluid is perturbed. In most situations the determination of the smallest-eigenvalue distribution and other allied quantities can be posed as the following problem in equilibrium statistical mechanics: what is the probability, $\mathrm{prob}(R)$, of finding a bubble of radius $R$ due to thermal fluctuations in the bulk of a fluid at temperature $T$? This important quantity measures the degree of correlation in the fluid. As the creation of a void requires energy, such events are relatively rare. Of particular interest is the behaviour of this probability when the bubble is large. A physical argument that gives a feel for the answer goes as follows: creating a bubble of radius $R$ in $d$ Euclidean dimensions requires a volume energy $\propto R^d$ and a surface energy $\propto R^{d-1}$. An application of the Boltzmann principle then suggests that

$$\mathrm{prob}(R) \sim \exp\left(-c_1\ R^d - c_2\ R^{d-1}\right) \tag{1}$$

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where $c_1$ and $c_2$, being respectively the volume and surface energies divided by $k_B T$, are positive constants independent of $R$. Equation (1) is valid for a bubble with a radius much larger than the coherence length. Thus, $\text{prob}(R) \sim e^{-\text{constant}\times R}$ in one dimension. For a fluid with short-range interactions (1) is expected to hold, but it should fail for a fluid with long-range interactions as is the case for a charged system, because the volume and surface energies are then no longer well defined. That such a consideration is pertinent to random-matrix problems is due to the appropriate measure in the space of $N \times N$ matrices consistent with symmetries. The invariant measure, according to a classification theorem of Dyson [6] is

$$P(M)\, dM = \exp\left[-\operatorname{tr} u(M)\right]\, dM = \prod_{1\leqslant a<b\leqslant N} |x_a - x_b|^\beta \prod_{c=1}^{N} \exp\left[-u(x_c)\right]\, dx_c\, dG$$

where $x_a$ are the eigenvalues and $dG$ is the Haar measure of the symmetry group that diagonalizes the matrices with orthogonal, unitary and symplectic symmetries, corresponding to $\beta = 1, 2, 4$, respectively.

The probability that an interval $J$ (a subset of $\mathbb{R}$) is free of eigenvalues, also known as the gap formation probability (GFP), is then

$$E_\beta[J] = \frac{\prod_{c=1}^{N} \left(\int_{\bar{J}} d\mu(x_c)\right) e^{-W(x_1,\ldots,x_N)}}{\prod_{c=1}^{N} \left(\int_{\bar{J}\cup J} d\mu(x_c)\right) e^{-W(x_1,\ldots,x_N)}} \tag{2}$$

where $\bar{J}$ is the compliment of $J$, $\bar{J} \cup J$ is natural support of the eigenvalues

$$d\mu(x) := \exp[-u(x)]dx$$

and

$$W(x_1, \ldots, x_N) := -\beta \sum_{1\leqslant a<b\leqslant N} \ln|x_a - x_b|. \tag{3}$$

Insight into this distribution can be gained by considering the $x$'s as the spatial coordinates of the particles and $u(x)$ can then be interpreted as a form of 'confining' potential holding together the logarithmically-repelling charged particles (all carrying charge $+1$) moving in one dimension. Under the combined influence of the confining and repelling forces the charges will attempt to settle down to an optimal configuration with small fluctuations around it. In the limit of large $N$ we can treat this system as a continuous fluid using techniques from electrostatics and thermodynamics. From the above consideration, the GFP can be written as

$$E_\beta[J] = e^{-\delta F[J]} \tag{4}$$

where

$$\delta F[J] := F[\bar{J}] - F[\bar{J} \cup J]$$

is the free energy of $N$ charges confined to region $\bar{J}$ minus the free energy of $N$ charges in the natural support of $d\mu$. As indicated above, the heuristic formula, equation (1), for the GFP fails for systems with long-range interaction. Instead, an alternative argument peculiar to charged systems which provides a qualitatively correct estimate for a large interval can be formulated as follows. An appropriate countercharge distribution is introduced externally to neutralize the region from which we wish to have the eigenvalues removed. If $F_Q$ is the free energy of the system with the addition of the countercharge ($Q$) and $F_{Q=0}$ the free energy of the undisturbed system then, according to electrostatics, $F_Q$ differs from $F_{Q=0}$ by an amount equal to the charging energy, i.e.

$$F_Q \sim F_0 + F_{\text{charging}} \tag{5}$$

where

$$F_{\text{charging}} = \frac{Q^2}{2C} \propto \mathcal{N}^2[J] \qquad (C = \text{capacitance})$$

$$= [\text{the total number of eigenvalues excluded from } J]^2 = \left[\int_J dx\ \sigma(x)\right]^2 \tag{6}$$

and $\sigma(x)$ is the eigenvalue/charge density†. Numerous examples supporting (6) can be found in [2, 3]. A further example that deals with the Laguerre ensemble is given in appendix A. We mention here that a screening theory of the charged fluid gives a physical justification [2] for equation (2), although we do not know of a proof of the general validity of this relationship.

In this paper we will use Dyson's continuum approximation, so let us first briefly outline what that entails. Dyson showed that the functional dependence of the free energy on the charge density is given by the functional

$$F[\sigma] := \int_K dx\ u(x)\sigma(x) - \frac{\beta}{2}\int_K dx \int_K dy\ \sigma(x)\ln|x-y|\sigma(y) + \left(1-\frac{\beta}{2}\right)\int_K dx\ \sigma(x)\ln[\sigma(x)] \tag{7}$$

where the fluid is confined to the region $K$ and the third term on the right-hand side of (7) came out of Dyson's careful analysis and will be referred to as the entropic term. Given this functional, then the equilibrium density $\sigma(x)$ is determined by minimization of $F[\sigma]$ with respect to the $\sigma(x)$ subject to the normalization constraint

$$\int_K \sigma(x)\ dx = N \tag{8}$$

$N$ being the number of charges, which is equal to the rank of the underlying matrices. Using Lagrange's method to handle this constrained minimization, which is equivalent to introducing a chemical potential, $A$, shows that the integral equation satisfied by the ground-state density is

$$u(x) - \beta\int_K dy\ \ \sigma(y)\ln|x-y| + \left(1-\frac{\beta}{2}\right)\ln\ \sigma(x) = A. \tag{9}$$

Therefore the free energy at equilibrium and with precisely $N$ charges contained in $K$ is

$$F[K] = \frac{1}{2}NA + \frac{1}{2}\int_K\ dx\ u(x)\ \sigma(x) - \left(1-\frac{\beta}{2}\right)\int_K\ dx\ \sigma(x)\ln\sigma(x) \tag{10}$$

the calculation of which for various different cases will form the core of this work.

This paper is organized as follows. In section 2 we revisit the Laguerre ensemble and compute the probability that the interval $[0, a]$ is free of eigenvalues by utilizing closed form expressions for certain integrals that were previously determined approximately and supply in a (double-) scaling limit the change in free energy including the entropic ($\beta \neq 2$) contribution. With this information we then present in section 3 the two-smallest-eigenvalue-distribution function for the Laguerre ensemble. In view of the application of random matrix

† Exact Painlevé analysis [7] for large $|J|$ and for $\beta = 2$, whenever such treatments are available, show that there is a sub-leading term $\sim \ln\mathcal{N}[J]$. This is not obtainable from the charging energy arguments. However, as was shown previously [1] and will be shown later for the Laguerre ensemble, the logarithmic correction is obtained in the continuum approximation. This is due to the fact that the density of the Laguerre ensemble is very large at the 'hard edge'.

ensembles to the theory of quantum transport in disordered solids (see [8] and references therein), this particular distribution function plays an important role in determining certain characteristics of the conductance of an ensemble of weakly disordered solids, for which the $x_j$ are the eigenvalues of the transmission matrices [13].

In section 4 we present our conclusions and discuss the extension of our method to the $m$-eigenvalue problem; details of the calculation of certain key integrals are given in the appendices, along with a note on the charging energy of the Laguerre ensemble.

## 2. The Laguerre potential revisited

The confining potential $u(x)$ for the Laguerre ensemble is

$$u(x) = x - \alpha \ln x \qquad x \geqslant 0 \quad \alpha > -1. \tag{11}$$

The change in free energy is computed as

$$\delta F = F[b, a] - F[b, 0] \tag{12}$$

where $F[b, a]$ is the free energy where all $N$ charges are supported on $a < x < b$. This would require the determination of the fluid density $\sigma(x)$ which satisfies the singular integral equation†

$$\frac{du}{dx} = \beta \, \mathrm{P} \int_a^b dy \, \frac{\sigma(y)}{x - y} \qquad a < x < b \tag{13}$$

the general solution of which for the ground state is‡

$$\sigma(x) = \frac{1}{\pi^2 \beta} \sqrt{\frac{b - x}{x - a}} \, \mathrm{P} \int_a^b \frac{dy}{y - x} \sqrt{\frac{y - a}{b - y}} \, \frac{du(y)}{dy} \qquad a < x < b. \tag{14}$$

For the Laguerre potential the fluid density is therefore

$$\sigma(x) = \frac{1}{\pi \beta} \sqrt{\frac{b - x}{x - a}} \left[ 1 - \frac{\alpha}{x} \sqrt{\frac{a}{b}} \right] \tag{15}$$

where we require that $\sqrt{ab} > \alpha$ so as to ensure that $\sigma(x)$ is positive semi-definite. The normalization condition now reads

$$N = \frac{b - a}{2\beta} + \frac{\alpha}{\beta} \left( \sqrt{\frac{a}{b}} - 1 \right) \tag{16}$$

which gives $b$ as a function of $N$ and $a$ via a solution of the following cubic equation (found from a slight rearrangement of (16)):

$$b^3 - 2(\nu + a) b^2 + (\nu + a) b - 4\alpha^2 = 0 \qquad \nu := 2\beta N + 2\alpha. \tag{17}$$

For sufficiently large $N$ this has a positive discriminant and therefore three real solutions, determinable by the trigonometric method [9]. Since it is a simple matter to convince oneself that $b$ decreases on increasing $a$ and that the leading-order contribution in $a$ should therefore be negative, it follows that the root which we require is

$$b = \frac{2(\nu + a)}{3} \left[ 1 + \cos \left( \frac{1}{3} \cos^{-1} \left[ -1 + \frac{54\alpha^2 a}{(\nu + \beta)^3} \right] - \frac{2\pi}{3} \right) \right]. \tag{18}$$

† This equation is found by differentiating equation (9) with respect to $x$, for $\beta = 2$. The $\beta \neq 2$ case will be considered later.

‡ $c/\sqrt{(b - x)(x - a)}$, where $c$ is a constant, solves the homogeneous part of (14), but including this would clearly increase the free energy and so it must be excluded from the solution for the ground-state density.

### 2.1. The free energy

Given $b$ and $\sigma(x)$ we can now determine the free energy which is given by the sum of the chemical potential contribution $F_{\text{chem}} := NA/2 = \big((\nu-\alpha)/4\beta\big)A$ and that due to interaction, i.e.

$$F_{\text{int}} := \frac{1}{2}\int_a^b \sigma(x)u(x)\,dx. \tag{19}$$

To compute the chemical potential we first drop the $\beta \neq 2$ correction in (9) (we will return to this term in section 2.3) and then evaluate it with $x \to b$, giving

$$A = b - \alpha\ln b - \frac{b-a}{2}\ln\left(\frac{b-a}{4e}\right) + \alpha\left[1-\sqrt{\frac{a}{a}}\,\right] + \alpha\sqrt{\frac{a}{b}}\,I_1\left(\frac{a}{b-a}\right) \tag{20}$$

where $I_1(x)$ is tabulated along with the other integrals $I_j(x)$, $j = 2,\ldots,4$, in table 1, whilst the details of their computation are confined to appendix B. Using table 1 we have

$$F_{\text{chem}} = \frac{\nu-\alpha}{4\beta}\left[a - \left(\frac{b-a}{2}\right)\ln\left(\frac{b-a}{4e}\right) - \alpha\sqrt{\frac{a}{b}}\ln\left(\frac{b-a}{4}\right) - 2\alpha\tanh^{-1}\sqrt{\frac{a}{b}}\,\right]. \tag{21}$$

The interaction contribution to the free energy is slightly more complicated:

$$F_{\text{int}} = \frac{(b-a)^2}{16\beta} + \frac{b-a}{4\beta}\big[a - \alpha\sqrt{a}b - \alpha\ln(b-a)\big] - \frac{\alpha(b-a)}{2\beta}I_3\left(\frac{a}{b-a}\right)$$
$$+\frac{\alpha^2\ln(b-a)}{2\beta}\left(1-\sqrt{\frac{a}{b}}\,\right) + \frac{\alpha^2}{2\beta}I_2\left(\frac{a}{b-a}\right) \tag{22}$$

where $I_2(x)$ and $I_3(x)$ are in table 1.

**Table 1.** Table of integrals. Note that all the results are for $x \geqslant 0$ and $y \geqslant 0$.

| Function | Integral representation | Closed form |
|---|---|---|
| $I_1(x)$ | $\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\frac{\ln(1-t)}{t+x}$ | $2\ln 2 + \sqrt{\frac{1+x}{x}}\left[\ln(1+x) - 2\tanh^{-1}\sqrt{\frac{x}{1+x}}\,\right]$ |
| $I_2(x)$ | $\sqrt{\frac{x}{1+x}}\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\frac{\ln(t+x)}{t+x}$ | $2\ln 2\sqrt{\frac{x}{1+x}} + \ln[4x(1+x)] - 2\left[1+\sqrt{\frac{x}{1+x}}\,\right]\tanh^{-1}\sqrt{\frac{x}{1+x}}$ |
| $I_3(x)$ | $\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\ln(t+x)$ | $-\frac{\ln(4e)}{2} - x + \sqrt{x(1+x)} + \ln[\sqrt{x}+\sqrt{1+x}]$ |
| $I_4(x)$ | $\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\frac{\ln t}{t+x}$ | $2\ln 2 + \sqrt{\frac{1+x}{x}}\left[\ln x - 2\tanh^{-1}\sqrt{\frac{x}{1+x}}\,\right]$ |
| $I_5(x,y)$ | $\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\frac{\ln(t+x)}{(t+y)}$ | $2\ln 2 - 2\sqrt{\frac{1+y}{y}}\tanh^{-1}\sqrt{\frac{y}{1+y}} - 2\tanh^{-1}\sqrt{\frac{x}{1+x}}$ $+\sqrt{\frac{1+y}{y}}\ln\left[x+y+2xy+2\sqrt{x(1+x)}\sqrt{y(1+y)}\right]$ |

### 2.2. The scaling limit of $F[b, a]$

From the expressions for $F_{\text{chem}}$ and $F_{\text{int}}$ we shall now obtain the free energy in the double scaling limit, i.e. the limit where $N$ (and therefore $\nu$) $\to \infty$ and simultaneously $a \to 0$, such that the combination $s := \nu a$ is finite. Under the above limit the free energy will become a function of $N$ plus a function of $s$ only; other terms of $\mathrm{O}(a^{\mu}/\nu^{\delta})$, $\mu, \delta > 0$ are discarded. Thus, expanding the free energies in $a$, we get

$$F[b,a] \sim F_0(N) + \frac{\alpha^2}{2\beta}\ln s + \frac{2\alpha}{\beta}\sqrt{s} + \frac{s}{2\beta} \qquad F_0(N) \sim -\frac{\beta}{2}N^2\ln N. \tag{23}$$

The GFP is thus given by

$$E_\beta(s) \sim \frac{\exp\left[-(s/2\beta) + (2\alpha/\beta)\sqrt{s}\right]}{s^{\alpha^2/2\beta}} \tag{24}$$

in agreement with previously obtained results [1, 7]. In [7], the logarithm of $E_2(s)$ is found to satisfy a particular Painléve III. Equation (24), for $\beta = 2$, agrees exactly with the main terms in the asymptotic expansion of obtained in [7].

### 2.3. The entropic contribution to the free energy for $\beta \neq 2$

The results so far obtained are strictly only true for $\beta = 2$, as for $\beta \neq 2$ there is an extra entropic contribution to the free energy, which we have ignored in the previous sections. However, if we assume that the factor $(1-\beta/2)$ is small, then to lowest order in this factor we can obtain the total free energy by adding in the term

$$F_{\mathrm{ent}} := \left(1 - \frac{\beta}{2}\right)\int_a^b \sigma(x)\,\ln\sigma(x)\,dx = \left(1 - \frac{\beta}{2}\right) f_{\mathrm{ent}} \tag{25}$$

to $F$, the density being as per equation (15), and, with $\Lambda := \left(\alpha/(b-a)\right)\sqrt{a/b}$ and $x := a/(b-a)$

$$f_{\mathrm{ent}} := -\ln(\pi\beta)N + f_{\mathrm{ent}}^{(1)} + f_{\mathrm{ent}}^{(2)} \tag{26}$$

where

$$f_{\mathrm{ent}}^{(1)} := \frac{b-a}{\pi\beta}\int_0^1 dt\,\sqrt{\frac{1-t}{t}}\left(1 - \frac{\Lambda}{t+x}\right)\ln\sqrt{\frac{1-t}{t}} \tag{27}$$

and

$$f_{\mathrm{ent}}^{(2)} := \frac{b-a}{\pi\beta}\int_0^1 dt\,\sqrt{\frac{1-t}{t}}\left(1 - \frac{\Lambda}{t+x}\right)\ln\left(1 - \frac{\Lambda}{t+x}\right). \tag{28}$$

We start with $f_{\mathrm{ent}}^{(1)}$ which can be straightforwardly reduced to

$$f_{\mathrm{ent}}^{(1)} = \frac{b-a}{\beta} - \frac{\Lambda(b-a)}{2\beta}\left[I_1(x) - I_4(x)\right] \tag{29}$$

$$= \frac{b-a}{\beta} + \frac{\alpha}{2\beta}\ln\left(\frac{a}{b}\right). \tag{30}$$

Thus, in the scaling limit, we find that

$$f_{\mathrm{ent}}^{(1)} \sim \frac{\nu}{\beta} + \frac{\alpha}{2\beta}\ln s + \mathrm{O}\left(\frac{s^{1/2}}{\nu}\right). \tag{31}$$

Concerning $f_{\mathrm{ent}}^{(2)}$, we first observe that $\Lambda/x < 1$, because of the constraint $\alpha < \sqrt{ab}$ imposed to ensure that the density is non-negative, which also makes certain that logarithm in $F_{\mathrm{ent}}$ never leads to imaginary terms. Thus, for $t \geqslant 0$, $\Lambda/(t+x) < 1$ and we can therefore evaluate $f_{\rm ent}^{(2)}$ by expanding the logarithm in (30) as a power series in $\Lambda/(t+x)$. In the scaling limit we find that

$$f_{\rm ent}^{(2)} \sim -\alpha + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{\alpha^{n+1}}{4^n\,[(n+1)!]^2}\left(1+\frac{1}{n}\right)\frac{1}{s^{n/2}} \tag{32}$$

the details of the derivation being in appendix C. We have, upon collecting equations (23), (31) and (32),

$$E_\beta(s) \sim c\,\frac{\exp\left[-(s/2\beta)+(2\alpha/\beta)\sqrt{s}\right]}{s^{(\alpha^2/2\beta)+\alpha(2-\beta)/4\beta}}\left[1+{\rm O}\left(\frac{1}{\sqrt{s}}\right)\right] \tag{33}$$

$c$ being a constant. Equation (33), valid for $\beta \neq 2$, now gives the asymptotic gap probability for arbitrary $\beta$ since our treatment is thermodynamical in nature. We note that equation (33) again agrees exactly with the main term of the asymptotic expansion found in [10] using the Aomoto generalization of the Selberg integral, which is only valid for integral $\beta$. It is well known from the variational computation of the free energy that the nonlinear term in the equilibrium equation (9), which arises from the $\beta \neq 2$ term, can be treated as a perturbation [5].

To conclude this section we note that from a probability argument we can also obtain the smallest-eigenvalue distribution $p_\beta^{(1)}(s)$, as it is related to $E_\beta(s)$ by†

$$p_\beta^{(1)}(s) = -\frac{dE_\beta(s)}{ds}. \tag{34}$$

Thus, from equations (33) and (34), we find that

$$p_\beta^{(1)} \sim \frac{c}{2\beta}\left(1-\frac{2\alpha}{s^{\frac{1}{2}}}+\left[\alpha+1-\frac{\beta}{2}\right]\frac{\alpha}{s}+\cdots\right)\frac{\exp\left[-(s/2\beta)+(2\alpha/\beta)\sqrt{s}\right]}{s^{(\alpha^2/2\beta)+\alpha(2-\beta)/4\beta}}. \tag{35}$$

## 3. The two-smallest-eigenvalue distribution

In this section we will calculate, within the continuum approximation, $p_\beta^{(2)}(x_1,x_2)$, the distribution function of the two smallest eigenvalues $x_1$ and $x_2$ ($x_2 \geqslant x_1$), given that the eigenvalues are distributed according to the joint probability distribution function

$$p(x_1,\ldots,x_N) = \frac{1}{Z}\prod_{a=1}^{N}{\rm e}^{-u(x_a)}\prod_{1\leqslant b<c\leqslant N}|x_b-x_c|^\beta \tag{36}$$

where $Z$ is the partition function. To calculate $p_\beta^{(2)}(x_1,x_2)$ we fix $x_1$ and $x_2$ and integrate $p(x_1,\ldots,x_N)$, with respect to $x_3,\ldots,x_N$ from $x_2$ to $\infty$, then

$$p_\beta^{(2)}(x_1,x_2) = \frac{N(N-1)}{2Z}|x_1-x_2|^\beta{\rm e}^{-u(x_1)-u(x_2)}\prod_{a=3}^{N}\left[\int_{x_2}^{\infty}dx_a\;{\rm e}^{-u(x_a)}|x_1-x_a|^\beta|x_2-x_a|^\beta\right]$$
$$\times\prod_{3\leqslant b<c\leqslant N}|x_b-x_c|^\beta. \tag{37}$$

† Letting $p(x_1,\ldots,x_n)$ denote the probability distribution function of the eigenvalues, then

$$E_\beta(x) = \int_x^\infty dx_1\cdots dx_n\;p(x_1,\ldots,x_n)$$

whilst

$$p_\beta^{(1)}(x) = n\int_x^\infty dx_2\cdots dx_n\;p(x,x_2,\ldots,x_n).$$

Equation (34) is thus demonstrated.

Shifting the integration variables using $x_a = t_a + x_2$ and relabelling the indices, we get

$$p_\beta^{(2)}(x_1, x_2) = \mathcal{C}|x_1 - x_2|^\beta \mathrm{e}^{-u(x_1)-u(x_2)} \prod_{a=1}^{N-2} \left[ \int_0^\infty dt_a \, \mathrm{e}^{-u(t_a+x_2)} \, |x_1 - x_2 - t_a|^\beta |t_a|^\beta \right] \times \prod_{1\leqslant b<c\leqslant N-2} |t_b - t_c|^\beta \tag{38}$$

where $\mathcal{C} = N(N-1)/2Z$. The multiple integral after the trivial factors will be evaluated using the continuum approximation described above. This is particularly interesting since the logarithmic (one-particle) factors already appear here. We expect, from the results obtained in section 2 that there will be an analogous logarithmic correction to the free energy. It is convenient to next transform to the centre-of-mass and difference coordinates, $\varepsilon = x_2 - x_1 (\geqslant 0)$ and $R = (x_1 + x_2)/2$ respectively, and then take $\bar{p}_\beta^2(\varepsilon, R) = p_\beta^{(2)}(x_1, x_2)$. Then

$$\bar{p}_\beta^{(2)}(\varepsilon, R) = \mathcal{C}\varepsilon^\beta \, \exp\left[-u\left(R - \frac{\varepsilon}{2}\right) - u\left(R + \frac{\varepsilon}{2}\right)\right] \times \int_0^\infty \left\{\prod_{a=1}^{N-2} dt_a\right\} \left[\mathrm{e}^{-u(t_a+R+\varepsilon/2)} \, (t_a(t_a+\varepsilon))^\beta \prod_{1\leqslant b<c\leqslant N-2} |t_b - t_c|^\beta\right]. \tag{39}$$

The multiple integral in (39) can be interpreted as the partition function for a charged fluid of $N-2$ of particles in the effective external potential $u_{\mathrm{eff}}(t; \varepsilon, R)$ defined by

$$u_{\mathrm{eff}}(t; \varepsilon, R) = u(t + R + \varepsilon/2) - \beta \ln[t(t+\varepsilon)]. \tag{40}$$

For reasons which we will explain later, the potential with $\alpha = 0$ is much simpler than the $\alpha \neq 0$ case, so we will first calculate the asymptotic expansion of $p_\beta^{(2)}$ for $\alpha = 0$. We therefore have

$$\bar{p}_\beta^{(2)}(\varepsilon, R) = \mathcal{C}\varepsilon^\beta \mathrm{e}^{-2R} \int_0^\infty \left\{\prod_{a=1}^{N-2} dx_a\right\} \left[\mathrm{e}^{-(x_a+R+\varepsilon/2)} \left[x_a(x_a+\varepsilon)^\beta\right] \prod_{1\leqslant b<c\leqslant N-2} |x_b - x_c|^\beta\right]. \tag{41}$$

The analogous integral equation for electrostatic equilibrium is

$$A = x + R + \frac{\varepsilon}{2} - \beta \ln x - \beta \ln(x+\varepsilon) - \beta \int_0^b dy \, \sigma(y) \ln|x - y| \tag{42}$$

where $A$ is the chemical potential, or

$$1 - \beta\left(\frac{1}{x} + \frac{1}{x+\varepsilon}\right) = \beta \, \mathrm{P} \int_0^b dy \, \frac{\sigma(y)}{y - x} \tag{43}$$

following the method outlined in section 2. The solution of (43) is

$$\sigma(x) = \frac{1}{\pi\beta}\sqrt{\frac{b-x}{x}} \left[1 - \frac{\beta}{x+\varepsilon}\sqrt{\frac{\varepsilon}{b+\varepsilon}}\,\right] \qquad 0 < x < b. \tag{44}$$

With the normalization $\int_0^b dx \, \sigma(x) = N - 2$, we find

$$N - 1 = \frac{b}{2\beta} + \sqrt{\frac{\varepsilon}{b+\varepsilon}} \tag{45}$$

which is equivalent to the cubic equation†

$$X^3 - 2(\nu+\varepsilon)X^2 + (\nu+\varepsilon)^2 X - 4\beta^2\varepsilon = 0 \qquad X := b+\varepsilon \quad \nu := 2\beta(N-1). \tag{46}$$

Again the discriminant of (46) is positive for sufficiently large $N$ and so we have three real roots. The appropriate solution is

$$b = \frac{2(\nu+\varepsilon)}{3}\left[1 + \cos\left(\frac{1}{3}\cos^{-1}\left(-1 + 54\beta^2\varepsilon/(\nu+\varepsilon)^3\right) - \frac{2\pi}{3}\right)\right] - \varepsilon. \tag{47}$$

The analogous free energy is

$$F = \frac{(\nu - 2\beta)A}{4\beta} + F_{\rm int} \tag{48}$$

where

$$F_{\rm int} := \frac{1}{2}\int_0^b dx\ \sigma(x) u_{\rm eff}(x;\varepsilon, R) \tag{49}$$

is the appropriate interaction energy.

### 3.1. *The chemical potential A*

To compute the chemical potential we take $x = b$ in (42), then

$$A = \frac{\ln(4e)}{2}b + R + \frac{\varepsilon}{2} - \beta\ln[b(b+\varepsilon)] - \frac{b\ln b}{2} + \beta\ln b\left[1 - \sqrt{\frac{x}{1+x}}\ \right] + \beta\sqrt{\frac{x}{1+x}} I_1(x) \tag{50}$$

defining $x := \varepsilon/b$. In the scaling limits $\nu \to \infty$ and $\varepsilon, R \to 0$ we find two suitable scaled variables: $S_1 := \nu\varepsilon$ and $S_2 := \nu(R + \varepsilon/2)$ (for reasons which will be made clear below), yielding

$$F_{\rm chem} \sim -\frac{\nu^2}{8\beta}\ln\left(\frac{\nu}{4e}\right) - \frac{\nu}{4}\ln(4e) + \frac{\beta}{2}\ln\nu - \frac{\sqrt{S_1}}{2} + \frac{S_2}{4\beta} + \mathrm{O}\left(\frac{S_1^{1/2}}{\nu}\right) \tag{51}$$

as the asymptotic expansion for $F_{\rm chem}$.

### 3.2. *The interaction energy*

In order to simplify the notation we first split up $F_{int}$ into the sum of two terms, $F_1$ and $F_{\ln}$, where

$$F_1 := \frac{1}{2\pi\beta}\int_0^1 dt\ \sqrt{\frac{1-t}{t}}\left(b - \sqrt{\frac{\varepsilon}{b+\varepsilon}}\frac{\beta}{t+x}\right)\left[bt + R + \frac{\varepsilon}{2} - 2\beta\ln b\right] \tag{52}$$

and

$$F_{\ln} := -\frac{1}{2\pi}\int_0^1 dt\ \sqrt{\frac{1-t}{t}}\left(b - \sqrt{\frac{\varepsilon}{b+\varepsilon}}\frac{\beta}{t+x}\right)\ \ln[t(t+x)]. \tag{53}$$

Evaluating these integrals and expanding the results we find that:

$$\begin{aligned} F_1 &= \frac{b^2}{16\beta} - \frac{b}{4}\sqrt{\frac{\varepsilon}{b+\varepsilon}} - \frac{\varepsilon}{2}\left(\sqrt{\frac{\varepsilon}{b+\varepsilon}} - 1\right) + \frac{N-2}{2}\left(R + \frac{\varepsilon}{2} - 2\beta\ln b\right) \\ &\sim \frac{\nu^2}{16\beta} - \frac{\nu\ln\nu}{2} - \frac{\sqrt{S_1}}{2} + \frac{S_2}{4\beta} + \mathrm{O}\left(\frac{S_1}{\nu}\right) \end{aligned} \tag{54}$$

† Recognising that we could obtain an *exact* solution for $b$ prompted us to look for closed-form expressions for the integrals and hence to re-examine the level-spacing distribution, the results for which we presented earlier.

and

$$F_{\ln} = \ln(2)b - \frac{b}{2}I_3(x) - \beta\ln b\left(\sqrt{\frac{x}{x+1}} - 1\right) + \frac{\beta}{2}I_2(x) + \frac{\beta}{2}\sqrt{\frac{x}{x+1}}I_4(x)$$

$$\sim \frac{1}{2}\left(\beta\ln 2 - \frac{1}{2}\ln(4e)\right)\nu - 2\beta\ln\nu + \beta\ln S_1 - \sqrt{S_1} + \mathrm{O}\left(\frac{S_1^{1/2}}{\nu}\right). \tag{55}$$

The asymptotic expansion of $F_{\text{int}}$ is thus

$$F_{\text{int}} \sim F_{\text{int}}^0(\nu) + \beta\ln S_1 - \frac{3}{2}\sqrt{S_1} + \frac{S_2}{4\beta}. \tag{56}$$

From equations (48), (51) and (56) we therefore find that asymptotically

$$F(\nu; S_1, S_2) \sim F^0(\nu) + \beta\ln(S_1) - 2\sqrt{S_1} + \frac{S_2}{2\beta}. \tag{57}$$

### 3.3. The entropic contribution to the free energy

As explained in section 2.3 when $\beta \neq 2$ there are additional terms in the free energy, the lowest order correction resembling an entropic contribution and is given by (25). We can once again split up $F_{\text{ent}}$ as per equation (26), with $f_{\text{ent}}^{(1)}$ and $f_{\text{ent}}^{(2)}$ as before, except for that the prefactor is changed according to $(b-a)/\pi\beta \to b/\pi\beta$ and we now take $\Lambda = (\beta/b)\sqrt{x/(1+x)}$. It follows that we now have

$$f_{\text{ent}}^{(1)} \sim \frac{\nu}{\beta} - \ln\nu + \frac{1}{2}\ln S_1 + \mathrm{O}\left(\frac{S_1^{1/2}}{\nu}\right) \tag{58}$$

and

$$f_{\text{ent}}^{(2)} \sim -1 + \sum_{n=1}^{\infty}\frac{(2n)!}{[(n+1)!]^2}\left(\frac{\beta}{4S_1^{1/2}}\right)^n \tag{59}$$

the last line of which is obtained in appendix C.

Thus, for $\alpha = 0$, but $\beta \neq 0$, the free energy is

$$F(\nu; S_1, S_2) \sim F_0(\nu) + \left[\beta + \frac{1}{2}\left(1 - \frac{\beta}{2}\right)\right]\ln S_1 - 2\sqrt{S_1} + \frac{S_2}{2\beta}$$

$$+ \left(1 - \frac{\beta}{2}\right)\sum_{n=1}^{\infty}\frac{(2n)!}{[(n+1)!]^2}\left(\frac{\beta}{4S_1^{1/2}}\right)^n. \tag{60}$$

### 3.4. The distribution for $\alpha \neq 0$

With the effective potential given by (40) and $\alpha \neq 0$, the charge density is given by

$$\sigma(x) = \frac{1}{\beta\pi}\sqrt{\frac{b-x}{x}}\left[1 - \frac{\alpha}{x+R+\varepsilon/2}\sqrt{\frac{R+\varepsilon/2}{b+R+\varepsilon/2}} - \frac{\beta}{x+\varepsilon}\sqrt{\frac{\varepsilon}{b+\varepsilon}}\right] \tag{61}$$

from which it follows that the normalization condition now reads

$$N - 1 = \frac{b}{2\beta} + \sqrt{\frac{\varepsilon}{b+\varepsilon}} - \frac{\alpha}{\beta}\left(1 - \sqrt{\frac{R+\varepsilon/2}{b+R+\varepsilon/2}}\right) \tag{62}$$

where, for ease of notation we have introduced $r := R + \varepsilon/2$. In contrast to the above calculations, we cannot re-express (62) in terms of a cubic equation, but only in terms of an eighth-order equation, which is why we previously described this problem as being more difficult than the others. However, if we define $\nu := 2\beta(N-1+\alpha/\beta)$, then we can rewrite (62) as

$$b = \nu - 2\alpha\sqrt{\frac{r}{b+r}} - 2\beta\sqrt{\frac{\varepsilon}{b+\varepsilon}} \tag{63}$$

which we can solve for $b$ as a power series in $\nu$, the result of the calculation being

$$b = \nu - 2\frac{\alpha r^{\frac{1}{2}} + \beta\varepsilon^{\frac{1}{2}}}{\nu^{\frac{1}{2}}} + \frac{\alpha^{\frac{3}{2}} + \beta\varepsilon^{\frac{3}{2}} - 2(\alpha+\beta)(\alpha r^{\frac{1}{2}} + \beta\varepsilon^{\frac{1}{2}})}{\nu^{\frac{3}{2}}} + \mathrm{O}\left(\nu^{-\frac{5}{2}}\right) \tag{64}$$

which for future reference we write in the compact form $b = \nu + \delta b$.

The chemical potential and interaction energy are calculated in precisely the same manner as before, so we evaluate the former at $x = b$ and split up the latter into the sum of $F_1$ and $F_{\mathrm{ln}}$. Introducing the further definitions

$$\mathcal{R} := \sqrt{\frac{r}{b+r}} \qquad \text{and} \qquad \mathcal{E} := \sqrt{\frac{\varepsilon}{b+\varepsilon}} \tag{65}$$

then the chemical potential is

$$\begin{aligned} A = {} & \tfrac{1}{2}\ln(4e)b + r - \alpha\ln(b+r) - \beta\ln[b(b+\varepsilon)] + \ln b\left(\alpha(1-\mathcal{R}) + \beta(1-\mathcal{E}) - \tfrac{1}{2}b\right) \\ & + \alpha\mathcal{R}I_1\left(\frac{r}{b}\right) + \beta\mathcal{E}I_1\left(\frac{\varepsilon}{b}\right) \qquad (66) \\ = {} & \tfrac{1}{2}\ln(4e)b + r - \left(\beta + \tfrac{1}{2}b + \alpha\mathcal{R} + \beta\mathcal{E}\right)\ln b + \ln(4)[\alpha\mathcal{R} + \beta\mathcal{E}] \\ & - 2(\alpha\tanh^{-1}\mathcal{R} + \beta\tanh^{-1}\mathcal{E}). \qquad (67) \end{aligned}$$

Asymptotically we therefore have

$$F_{\mathrm{chem}} \sim F^0_{\mathrm{chem}}(\nu) - \frac{\sqrt{S_1}}{2} - \frac{\alpha\sqrt{S_2}}{2\beta} + \frac{S_2}{4\beta}. \tag{68}$$

As for the interaction energy, it is simple to arrive at

$$F_1 = \frac{b^2}{16\beta} - \frac{b}{4\beta}(\alpha\mathcal{R} + \beta\mathcal{E}) - \frac{\alpha r(1-\mathcal{R}) + \beta\varepsilon(1-\mathcal{E})}{2\beta} + \frac{N-2}{2}\left(r - (\alpha+2\beta)\ln b\right) \tag{69}$$

and thus

$$F_1 \sim F^0_1(\nu) - \frac{\sqrt{S_1}}{2} - \frac{\alpha\sqrt{S_2}}{2\beta} + \frac{S_2}{4\beta}. \tag{70}$$

In full, the expression for $F_{\mathrm{ln}}$ is

$$\begin{aligned} F_{\mathrm{ln}} = {} & -\left(1 + \frac{\alpha}{\beta}\right)\ln(b)\left(\frac{b}{2} - \alpha(1-\mathcal{R}) - \beta(1-\mathcal{E})\right) \\ & + \frac{b}{2\beta}\left(\frac{\beta}{2}\ln(4e) - \alpha I_3\left(\frac{r}{b}\right) - \beta I_3\left(\frac{\varepsilon}{b}\right)\right) + \frac{\beta}{2}\left[\mathcal{R}I_2\left(\frac{\varepsilon}{b}\right) + \mathcal{E}I_4\left(\frac{r}{b}\right)\right] \\ & + \frac{\alpha}{2}\left[\mathcal{R}I_4\left(\frac{r}{b}\right) + \mathcal{R}I_5\left(\frac{\varepsilon}{b}, \frac{r}{b}\right) + \mathcal{E}I_5\left(\frac{r}{b}, \frac{\varepsilon}{b}\right)\right] + \frac{\alpha^2}{2\beta}I_2\left(\frac{r}{b}\right) \qquad (71) \end{aligned}$$

which yields the asymptotic expansion

$$F_{\mathrm{ln}} \sim F^0_{\mathrm{ln}}(\nu) - \sqrt{S_1} + \beta\ln(S_1) - \frac{\alpha}{\beta}\sqrt{S_2} + \frac{\alpha}{2}\left(1 + \frac{\alpha}{\beta}\right)\ln(S_2) + 2\alpha\ln(\sqrt{S_1} + \sqrt{S_2}). \tag{72}$$

So, from equations (68), (70) and (72), the asymptotic expansion of the free energy for non-zero $\alpha$ is

$$F(\nu; S_1, S_2) \sim F^0(\nu) + \beta \ln(S_1) - 2\left(\sqrt{S_1} + \frac{\alpha}{\beta}\sqrt{S_2}\right) + \frac{S_2}{2\beta} + \frac{\alpha}{2}\left(1 + \frac{\alpha}{\beta}\right)\ln(S_2)$$
$$+ 2\alpha \ln(\sqrt{S_1} + \sqrt{S_2}). \tag{73}$$

### 3.5. The distribution of the two smallest eigenvalues

To finally obtain the asymptotic result for the two-smallest-eigenvalue distribution we first introduce $s_i := \nu x_i$, i.e. the double-scaled variables of the two smallest eigenvalues. Next, we define $P_\beta^{(2)}(s_1, s_2) = \lim_{\nu\to\infty} p_\beta^{(2)}(s_1/\nu, s_2/\nu)$, which is the double-scaled asymptotic of the distribution. From (39) and the observation that $S_1 = s_2 - s_1$ and $S_2 = s_2$, we therefore have

$$P_\beta^{(2)}(\alpha; s_1, s_2) = \mathcal{C} \exp\left(\beta \ln(s_2 - s_1) - F(\nu; s_2 - s_1, s_2)\right). \tag{74}$$

Explicitly, the asymptotic distribution with $\alpha = 0$, but $\beta \neq 2$, is

$$P_\beta^{(2)}(0; s_1, s_2) = \mathcal{C}_\nu \exp\Bigg[-\frac{1}{2}\left(1 - \frac{\beta}{2}\right)\ln(s_2 - s_1) + 2\sqrt{s_2 - s_1} - \frac{s_2}{2\beta}$$
$$-\left(1 - \frac{\beta}{2}\right)\sum_{n=1}^{\infty} \frac{(2n)!}{[(n+1)!]^2}\left(\frac{\beta}{4(s_2 - s_1)^{\frac{1}{2}}}\right)^n\Bigg] \tag{75}$$

where $\mathcal{C}_\nu$ represents all the $s_i$-independent terms.

Similarly, with $\alpha \neq 0$ and $\beta = 2$, from equations (73) and (74), we then have

$$P_2^{(2)}(\alpha; s_1, s_2) = \mathcal{C}_\nu \exp\Big[-\tfrac{1}{4}s_2 + 2\sqrt{s_2 - s_1} + \alpha\left(\sqrt{s_2} - \tfrac{1}{2}(1 + \tfrac{1}{2}\alpha)\ln(s_2)\right)$$
$$-2\alpha \ln\left(\sqrt{s_2 - s_1} + \sqrt{s_2}\right)\Big]. \tag{76}$$

### 3.6. A comparison of the continuum approach with exact results

Exact results for the distribution of the two smallest eigenvalues have been obtained by Forrester and Huges [13] for the unitary Laguerre ensemble with $\alpha = \ell$, where $\ell$ is either zero or a positive integer. We are thus able to check our results and will do so for the cases $\alpha = 0$ and $\alpha = 1$. In order to proceed we first note that in terms of the notation in [13], we have $a = 1$ and $m = N$. Secondly, the variables $s_i$ introduced in the previous sub-section are, in the limit of large $N$, the same as the $s_i$ used in [13]; we are now ready to check our results. Forrester and Huges have shown that an exact expression for the scaled asymptotic distribution is

$$P_2^{(2)}(\ell; s_1, s_2) = 2^{-4} e^{-(s_2/4) + \ell \ln(s_2/s_1)}$$
$$\times \det \begin{bmatrix} \left[I_{j-k+2}(s_2^{1/2})\right]_{\substack{j=1,\ldots,\ell \\ k=1,\ldots,\ell+2}} \\ \left[\left(\frac{s_2 - s_1}{s_2}\right)^{\frac{1}{2}(k-j)} I_{j-k+2}(\sqrt{s_2 - s_1})\right]_{\substack{j=1,2 \\ k=1,\ldots,\ell+2}} \end{bmatrix} \tag{77}$$

which requires us to compute the determinant of an $(\ell+2)\times(\ell+2)$ matrix of modified Bessel functions of the first kind, $I_\nu(x)$. Note that these are not to be confused with those in table 1.

For $\alpha = 0$ and $\alpha = 1$, we thus find that

$$P^{(2)}(0; s_1, s_2) = \tfrac{1}{16}\mathrm{e}^{-s_2/4}\left[I_2\left(\sqrt{s_2-s_1}\right)^2 - I_1\left(\sqrt{s_2-s_1}\right)I_3\left(\sqrt{s_2-s_1}\right)\right] \tag{78}$$

and

$$\begin{aligned} P^{(2)}(1; s_1, s_2) = \frac{1}{16}\mathrm{e}^{-(s_2/4)+\ln(s_2/s_1)}\Bigg\{&\left(\frac{s_2-s_1}{s_2}\right)I_2(s_2) \\ &\times\left[I_1\left(\sqrt{s_2-s_1}\right)^2 - I_0\left(\sqrt{s_2-s_1}\right)I_2\left(\sqrt{s_2-s_1}\right)\right] - \sqrt{\frac{s_2-s_1}{s_2}}I_1(\sqrt{s_2}) \\ &\times\left[I_1\left(\sqrt{s_2-s_1}\right)I_2\left(\sqrt{s_2-s_1}\right) - I_0\left(\sqrt{s_2-s_1}\right)I_3\left(\sqrt{s_2-s_1}\right)\right] \\ &+ I_0(\sqrt{s_2})\left[I_2\left(\sqrt{s_2-s_1}\right)^2 + I_1\left(\sqrt{s_2-s_1}\right)I_3\left(\sqrt{s_2-s_1}\right)\right]\Bigg\} \end{aligned} \tag{79}$$

respectively. We claim that the Coulomb fluid gives the asymptotic limit of the given distribution, i.e. the distribution in the limit of $\sqrt{s_2-s_1}, \sqrt{s_2} \gg 1$. In the limit of large $x$, the modified Bessel functions have the asymptotic expansion

$$I_\nu(x) \sim \frac{\mathrm{e}^{-x}}{\sqrt{2\pi x}}\left(1 + \frac{4\nu^2-1}{8x} + \mathrm{O}(x^{-2})\right). \tag{80}$$

The asymptotic expansion of the given distribution is therefore

$$P_2^{(2)}(0; s_1, s_2) \sim \frac{\mathrm{e}^{-(s_2/4)+2\sqrt{s_2-s_1}}}{32\pi}\left[\frac{1}{s_2-s_1} + \mathrm{O}\left((s_2-s_1)^{-2}\right)\right] \tag{81}$$

for $\alpha = 0$, whilst for $\alpha = 1$ we have

$$P_2^{(2)}(1; s_1, s_2) \sim \frac{\mathrm{e}^{-s_2/4}}{16(2\pi)^{\frac{3}{2}}}\mathrm{e}^{2\sqrt{s_2-s_1}+\sqrt{s_2}}s_2^{-\frac{5}{4}}\left[1+\cdots\right]. \tag{82}$$

From the Coulomb-fluid approximation we get

$$P_2^{(2)}(0; s_1, s_2) \sim \mathrm{e}^{-(s_2/4)+2\sqrt{s_2-s_1}} \tag{83}$$

and

$$\begin{aligned} P_2^{(2)}(1; s_1, s_2) \sim &\exp\left(-\tfrac{1}{4}s_2 + 2\sqrt{s_2-s_1} + \sqrt{s_2}\right) \\ &\times\exp\left(-\tfrac{3}{4}\ln(s_2) - 2\ln\left(\sqrt{s_2-s_1}+\sqrt{s_2}\right)\right). \end{aligned} \tag{84}$$

Comparison of equations (81) and (83) and equations (82) and (84) reveals that the continuum approximation successfully captures the dominant term in the asymptotic expansion of the Bessel functions, i.e. the term $\mathrm{e}^{-x}$, although the logarithmic factors are not precisely reproduced.

## 4. Conclusions

In this paper we have used the continuum approximation to calculate the GFP, and hence the smallest-eigenvalue distribution, and the two-smallest-eigenvalue distribution of the Laguerre ensemble of random matrices.

The GFP has been the subject of a previous paper [1], but here we presented a calculation in which key intermediate results were given in closed form, in contrast to the simple power series which we had formerly obtained.

The two-smallest-eigenvalue distribution was new and demonstrates how it is possible to extend the continuum approximation to deal with higher-order correlation functions. The two essential differences in the calculation being: (i) the normalization condition is that there should be $N-2$ charges in the interval, and (ii) the set of the smallest eigenvalues modify the potential in which the remaining charges are placed, thereby leading to an effective potential instead of the plain Laguerre potential. Exact results for this distribution are available for $\beta = 2$ and $\alpha = 0$ or a positive integer and we have shown that the continuum approximation gives the leading-order terms in the asymptotic expansion of this distribution for $\alpha = 0$ and 1. We therefore feel justified in claiming that our result not only gives the dominant terms for $\alpha = 2, 3, \ldots$ correctly, but that the leading-order factor in the asymptotic expansion of this distribution *for all* $\alpha$ and $\beta$ is given by

$$P_\beta^{(2)}(\alpha; s_1, s_2) \sim \exp\left[-\frac{s_2}{2\beta} + 2\sqrt{s_2 - s_1} + \alpha\sqrt{s_2}\right]\varphi(s_1, s_2) \tag{85}$$

where $\varphi(s_1, s_2)$ is an unknown function of $s_1$ and $s_2$. The present computation suggests that

$$\varphi(s_1, s_2) = s_2^{-\alpha(1/2+\alpha/4)} \left(\sqrt{s_2 - s_1} + \sqrt{s_2}\right)^{-2\alpha}$$

and is asymptotic to $s_2^{-\alpha(3/2+\alpha/4)}$, for $s_2 \gg s_1$. This compares favourably with $s_2^{-5/4}$ for $\alpha = 1$.

Thus, although the Coulomb fluid is limited to the dominant factor, the method is somewhat simpler than the known method of deriving the distribution exactly [13], and also has the advantage of being applicable, we believe, to the full range of parameters of interest.

Since we have shown that the Coulomb-fluid approximation can be applied rather successfully to problems regarding the smallest and the two-smallest eigenvalues, it is only natural to ask whether we can use it examine the $m$-smallest-eigenvalue distribution. If $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_m$ denote the $m$ smallest eigenvalues, then, from equation (36), the $m$-smallest-eigenvalue distribution can be written in the form

$$p_\beta^{(m)}(x_1, \ldots, x_m) = \mathcal{C}_\beta^{(m)} \prod_{1\leqslant a<b\leqslant m} |x_a - x_b|^\beta \exp\left(-\sum_{c=1}^{m} u(x_c)\right)\mathcal{D}_\beta^{(m)} \tag{86}$$

where $\mathcal{C}_\beta^{(m)}$ is the normalization constant and

$$\mathcal{D}_\beta^{(m)} := \int_{x_m}^\infty dx_{m+1} \cdots \int_{x_m}^\infty dx_N \left[\exp\left(-\sum_{a=m+1}^{N} u(x_a)\right) \prod_{1\leqslant b\leqslant m, m+1\leqslant c\leqslant N} |x_b - x_c|^\beta \right.$$
$$\left. \times \prod_{m+1\leqslant d<e\leqslant N} |x_d - x_e|^\beta\right]. \tag{87}$$

Introducing the change of variable $t_{a-m} = x_a - x_m$, we have thus transformed the problem of calculating $p_\beta^{(m)}$ to the problem of determining $\mathcal{D}_\beta^{(m)}$, where

$$\mathcal{D}_\beta^{(m)} = \int_0^\infty dt_1 \cdots \int_0^\infty dt_{N-m} \ \exp\left(-\sum_{a=1}^{N-m} u_{\text{eff}}(t_a; x_1, \ldots, x_m)\right) \prod_{1\leqslant b<c\leqslant N-m} |t_b - t_c|^\beta \quad (88)$$

the effective potential now being

$$u_{\text{eff}}(t; x_1, \ldots, x_m) := u\,(t + x_m) - \beta \sum_{a=1}^{m} \ln|t + x_m - x_a|\,. \quad (89)$$

We propose that $\mathcal{D}_\beta^{(m)}$ can be calculated asymptotically using the Coulomb-fluid approximation with the normalization being set equal to $N - m$; we will not pursue this problem any further here.

With the application of the Coulomb-fluid method introduced by Dyson [5] and the aid of thermodynamics and electrostatics, we have determined certain eigenvalue distribution functions of the Laguerre ensemble, namely the probability distribution for the smallest and two smallest eigenvalues. We have found that the asymptotic probability distribution agrees very well with those obtained using more specialized techniques [7, 10, 13]. In the case of the smallest-eigenvalue distribution, the results we have obtained generalizes to non-integer values for $\beta$, and when comparison is possible it is found that *all* the main terms in the asymptotic expansion, including the pre-exponential factor are in precise agreement with the exact asymptotic results. A physical explanation for the high accuracy of the Coulomb-fluid method is that the eigenvalue density of the Laguerre ensemble is largely weighted near the origin, this is reflected in the square-root singularity in the density. As the continuum approximation uses techniques from macroscopic physics, we may expect an accurate result for computing the change in the free energy by excluding the charged fluid away from the origin, since the density is rather large there. This also points towards an explanation for the less accurate results obtained for the two-smallest-eigenvalue distribution. In computing the above quantity, the interval of exclusion appropriate is somewhat larger then the smallest eigenvalue case, causing disagreement in the pre-exponential factor. We remark here that a similar disagreement was also found in computing the gap probability of the Dyson circular ensemble, however, the error made in missing the logarithmic term is $\mathrm{O}(\ln N/N^2)$. This term can be restored by using (24) and a mapping found in [7], relating the Laguerre ensemble and the Gaussian ensemble, see [1] for a detailed discussion.

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## Appendix A. The charging energy of the Laguerre ensemble

The integral equation for the density, equation (9), for $\beta = 2$ and with the specialization to the Laguerre ensemble $(u(x) = x - \alpha \ln x)$ reads

$$x - \alpha \ln x - \beta \int_0^b dy\, \sigma(y) \ln|x - y| = A \qquad 0 < x < b. \quad (A1)$$

In addition to regular solutions, equation (A1) also admits solutions that describe point charges. Including these would produce infinite self-energy thus causing the total free energy to diverge. Bearing in mind that the GFP is expressed as the difference of two free energies (see equation (4)), the self-energy divergences cancel. With the point charge at the origin superimposed on the regular solution, equation (A1) has the following solution:

$$\sigma(x) = -\frac{\alpha}{\beta}\delta_+(x) + \frac{1}{\pi\beta}\sqrt{\frac{b-x}{x}} \qquad 0 \leqslant x < b. \tag{A2}$$

Observe that the point charge carries a weight of order unity while the normalization condition $\int_0^b \sigma(x)\,dx = N$, is almost exhausted by the regular distribution. From the normalization condition we find that $b = 2\beta N + \alpha$. A simple calculation shows that the number of eigenvalues excluded from the interval, $[0, a]$, $(a < b)$ is

$$\mathcal{N}[a] = \int_0^a \sigma(x)\,dx = -\frac{\alpha}{2\beta} + \frac{1}{\pi\beta}\left[\sqrt{a(b-a)} + b\tan^{-1}\sqrt{\frac{a}{b-a}}\right]. \tag{A3}$$

In the scaling limit $b \to \infty$ and $a \to 0$ such that $ab = s$ is finite, we find

$$\mathcal{N}[a] \sim -\frac{\alpha}{2\beta} + \frac{2}{\pi\beta}\sqrt{ab} + \mathrm{O}\left(\frac{a^{3/2}}{b^{1/2}}\right) \tag{A4}$$

and

$$E_\beta(s) \sim \exp\left[-a_1 s + a_2\sqrt{s}\right] \tag{A5}$$

where $a_1$ and $a_2$ are constants independent of $s$. Equation (A5) is in qualitative agreement with (20). As noted in the introduction the charging energy argument does not supply the logarithmic correction.

## Appendix B. Evaluation of the integrals

The integrals $I_1$, $I_2$ and $I_4$ are all evaluated by the same method, whilst $I_3$ is evaluated by elementary means and is given later. Firstly, we use the identity $h^\lambda \ln h = \partial_\lambda\, h^\lambda$, where $\partial_\lambda = \partial/\partial\lambda$, which for the integrands of interest yields an integral proportional to the Gauss hypergeometric function [11]. More precisely, we have

$$I_1(x) = \frac{1}{\pi}\partial_\lambda\left[B\left(\lambda, \frac{1}{2}\right) {}_2F_1\left(1, \frac{1}{2}; \lambda + \frac{1}{2}; -\frac{1}{x}\right)\right]\bigg|_{\lambda=\frac{1}{2}} \tag{B1}$$

$$I_2(x) = \frac{1}{2}\sqrt{\frac{x}{x+1}}\,\partial_\lambda\left[x^\lambda {}_2F_1\left(-\lambda, \frac{1}{2}; 2; -\frac{1}{x}\right)\right]\bigg|_{\lambda=-1} \tag{B2}$$

and

$$I_4(x) = \frac{1}{\pi x}\partial_\lambda\left[B\left(\lambda, \frac{1}{2}\right) {}_2F_1\left(1, \lambda; \frac{3}{2} + \lambda; -\frac{1}{x}\right)\right]\bigg|_{\lambda=\frac{1}{2}}. \tag{B3}$$

Next, we apply the transformation formulae [12] which send ${}_2F_1(.,.,.,.,1/x)$ to ${}_2F_1(.,.,.,.,x)$ to get a convergent hypergeometric series. Since this step and those which follow do not essentially differ for the three integrals, we only display the calculation for $I_2(x)$. The transformation formula used is equation (9.1322) of [11], from which

$$I_2(x) = \sqrt{\frac{x}{1+x}}\left[2\ln\left(\frac{2}{e}\right) - i_1(x) + \frac{i_2(x)}{\sqrt{x}}\right] + \ln(4x) \tag{B4}$$

where

$$i_1(x) := \partial_\lambda {}_2F_1\left(1-\lambda, -\lambda; \tfrac{3}{2} - \lambda; -x\right)\Big|_{\lambda=0} \tag{B5}$$

and

$$i_2(x) := \partial_\lambda \left[ (1+x)^{\lambda+\frac{1}{2}} {}_2F_1\left(\lambda, \lambda+1; \lambda+\tfrac{1}{2}; -x\right)\right]\Big|_{\lambda=0}. \tag{B6}$$

To compute $i_1$ and $i_2$ we use the series representation of ${}_2F_1$, which is differentiated to give

$$i_1(x) = -\sum_{n=1}^{\infty} B\left(\frac{3}{2}, n\right)(-x)^n \tag{B7}$$

and

$$i_2(x) = \sqrt{1+x}\left[\ln(1+x) + \sum_{n=1}^{\infty} B\left(\frac{1}{2}, n\right)(-x)^n\right]. \tag{B8}$$

Using the integral representation,

$$B(x, y) = \int_0^1 dt\ t^{x-1}(1-t)^{y-1} \qquad \mathrm{Re}\, x, \mathrm{Re}\, y > 0$$

of the Beta function, followed by exchanging the order of integration and summation, we find with the aid of $\sum_{n=0}^{\infty}(-xt)^n = (1+xt)^{-1},\ |xt| < 1$, that

$$i_1(x) = x\int_0^1 dt\ \frac{\sqrt{1-t}}{1+xt} \tag{B9}$$

and

$$i_2(x) = \sqrt{1+x}\left[\ln(1+x) - x\int_0^1 \frac{dt}{\sqrt{1-t}(1+xt)}\right]. \tag{B10}$$

The integrals in equation (B9) and (B10) are expressed in terms of elementary functions

$$i_1(x) = -2 + 2\sqrt{\frac{1+x}{x}}\tanh^{-1}\sqrt{\frac{x}{1+x}} \tag{B11}$$

and

$$i_2(x) = \sqrt{1+x}\left[\ln(1+x) - 2\sqrt{\frac{x}{1+x}}\tanh^{-1}\sqrt{\frac{x}{1+x}}\right] \tag{B12}$$

and thus the result for $I_2(x)$ follows.

To determine $I_3(x)$, we first compute $I_3'(x)$. The $t$-integration is then elementary, giving

$$I_3'(x) = -1 + \sqrt{\frac{1+x}{x}}. \tag{B13}$$

We find by integrating (B13) and supplying a constant that

$$I_3(x) = I_3(0) + \int_0^x \sqrt{\frac{1+y}{y}}\, dy = -\frac{\ln(4e)}{2} - x + \sqrt{x(1+x)} + \ln\left(\sqrt{x} + \sqrt{1+x}\right). \tag{B14}$$

The function $I_5(x, y)$ requires a little more effort to obtain its closed form. We begin by writing the factor $\ln(t+x) = \ln(t) + P\int_0^x du/(u+t)$ and thereby obtain

$$I_5(x, y) = I_4(y) + \mathrm{P}\int_0^x \frac{du}{u-y}\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\left(\frac{1}{t+y} - \frac{1}{t+u}\right). \tag{B15}$$

The $t$-integration is easily performed and leads to

$$I_5(x, y) = I_4(y) + \mathrm{P}\int_0^x \frac{du}{u-y}\left(\sqrt{\frac{1+y}{y}} - \sqrt{\frac{1+u}{u}}\right). \tag{B16}$$

Given that

$$P\int_0^x \frac{du}{u-y}\sqrt{\frac{1+u}{u}} = \ln\left[1+2x+2\frac{x}{1+x}\right] + \sqrt{\frac{1+y}{y}}\left[\ln(x-y) + \ln\left(-x-y-2xy - 2\sqrt{x(1+x)}\sqrt{y(1+y)}\right)\right] \tag{B17}$$

then the result for $I_5$ shown in table 1 follows easily.

## Appendix C. Entropic contribution to the free energy

The entropic term $f_{\text{ent}}^{(2)}$ for the smallest eigenvalue distribution is given by

$$f_{\text{ent}}^{(2)} = \frac{b-a}{\beta}\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\left(1-\frac{\Lambda}{t+x}\right)\ln\left(1-\frac{\Lambda}{t+x}\right) \tag{C1}$$

where $\Lambda := (\alpha/(b-a))\sqrt{a/b}$. We could evaluate this in closed form, but this would require a method different to that in appendix B and as we are most interested in the scaling-limit form it is more sensible to focus our attention on that. To this end we proceed by first expanding the logarithm in (C1) as a Taylor series in $\Lambda/(t+x)$, since $\sqrt{ab} > \alpha$ (the positivity condition on the density) implies that $\Lambda/(t+x) < 1$ for $t > 0$. Doing so gives

$$f_{\text{ent}}^{(2)} = \frac{1}{\beta}\left[-g_1 + \sum_{n=1}^{\infty}\frac{g_{n+1}}{n(n+1)}\right] \qquad g_1 = \alpha\left(\sqrt{\frac{a}{b}}-1\right) \tag{C2}$$

where

$$g_n = \alpha\sqrt{\frac{a}{b}}\Lambda^{n-1}\int_0^1 \frac{dt}{\pi}\sqrt{\frac{1-t}{t}}\left(\frac{1}{t+x}\right)^n = \alpha\sqrt{\frac{a}{b}}\frac{(-\Lambda)^{n-1}}{(n-1)!}\left(\frac{d}{dx}\right)^{n-1}\sqrt{\frac{1+x}{x}} \qquad n \geqslant 2. \tag{C3}$$

Writing $(d/dx)^m\sqrt{1+x} = u_m(1+x)^{\frac{1}{2}-m}$ and $(d/dx)^m x^{-\frac{1}{2}} = d_m x^{-\frac{1}{2}-m}$, then an application of the Liebnitz rule gives

$$g_n = (-1)^{n-1}\alpha^n\left(\frac{a}{b}\right)^{n/2}\sum_{p=0}^{n-1}\frac{u_p d_{n-1-p}}{p!(n-1-p)!}\, b^{\frac{1}{2}-p}\, a^{p-n-\frac{1}{2}}. \tag{C4}$$

In the scaling limit we see that the only significant term in (C4) is the $p = 0$ one. Since $u_0 = 1$ (trivially) and $d_m = (-1)^m(2m)!/(2^{2m}m!)$, then

$$g_n = \frac{\alpha^n(2n-2)!}{2^{2n-2}[(n-1)!]^2}\,\frac{1}{(ab)^{(n-1)/2}} + \mathrm{O}\left(\frac{a}{b(ab)^{(n-1)/2}}\right) \tag{C5}$$

and therefore

$$f_{\text{ent}}^{(2)} \sim -\alpha + \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{\alpha^{n+1}(2n)!}{4^n[(n+1)!]^2}\left(1+\frac{1}{n}\right)\frac{1}{s^{n/2}}. \tag{C6}$$

The analogous contribution in the two-smallest eigenvalue problem only differs from the above in that the prefactor is $b/\beta$, as opposed to $(b-a)/\beta$, $x = \varepsilon/b$ and $\Lambda = (\beta/b)\sqrt{(b+\varepsilon)/\varepsilon}$. We therefore now have

$$f_{\text{ent}}^{(2)} \sim -1 + \sum_{n=1}^{\infty}\frac{(2n)!}{4^n[(n+1)!]^2}\left(1+\frac{1}{n}\right)\left(\frac{\beta}{\sqrt{s}}\right)^n. \tag{C7}$$

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